# Parameter counting
# for neutrino mixing

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## Abstract

The content of physical masses, mixing angles and $CP$ violating phases in the lepton sector of extended standard model, both renormalizable and non-renormalizable, with arbitrary numbers of the singlet and left-handed doublet neutrinos is systematically analysed in the weak basis.

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# 1 Introduction

The mixing of quarks in the minimal Standard Model (SM) of strong and electroweak interactions is now well understood. It is described by the Cabibbo-Kobayashi-Maskawa (CKM) unitary mixing matrix for the charged currents [1], the neutral ones and Yukawa interactions being flavour conserving. As for the lepton sector, the SM exhibits here an extremely simple and economic structure. It encounters just three physical parameters, the charged lepton masses, and predicts no flavour and $CP$ violation. But it has been widely recognized that the inclusion of (iso)singlet neutrinos and/or neutrino masses in the SM would result in the lepton mixing and flavour violation with all the related phenomena such as $CP$ violation, neutrino oscillations [2], etc. (as a recent review see, e.g., ref. [3]).

There are two principal differences between the lepton and quark mixings. First, the number of singlet neutrinos relative to that of (iso)doublet ones is not restricted by the chiral anomalies and can be arbitrary. Second, the Majorana masses for neutrinos are possible in addition to the Dirac ones. This complicates inevitably the proper SM extensions and proliferates free parameters. Hence, an immediate problem arises how to extract the physical parameters, to separate masses, mixing angles and $CP$ violating phases among them, as well as to conveniently parametrize the mixing matrices. There have been many studies of the related topics. The case with an arbitrary number of the left-handed doublet neutrinos but without singlet ones was considered in ref. [4], the one with equal arbitrary numbers of the singlet and doublet neutrinos in ref. [5] and a general case with arbitrary numbers of both types of neutrinos in ref. [6]. In particular, the last case with only Dirac masses was studied in ref. [7]. Traditionally, all the previous investigations were carried out by an explicit construction in the mass basis.

In the present paper an alternative approach to the lepton parameter counting is adopted. It is formulated in the weak basis entirely through symmetry properties of the model before the spontaneous symmetry breaking [8]. In these terms, all the possible parameter space configurations of the SM extended with arbitrary numbers of the singlet and left-handed doublet neutrinos are systematically analysed. Both renormalizable and non-

renormalizable extensions of the SM, among them the pure Dirac and pure Majorana cases, are considered. In a consistent fashion, the known results on the lepton parameter counting for the SM general extensions are recovered, and the new ones for the renormalizable extensions are obtained. The relation between the two countings is clarified. The results on parameter counting for neutrino mixing are summarized in tables in what follows.[1]

## 2    Renormalizable extensions

**(i) Arbitrary case**    The most general renormalizable $SU(2)_{\mathrm{W}} \times U(1)_{\mathrm{Y}}$ invariant lepton Lagrangian of the SM extended by the right-handed neutrinos reads

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\begin{aligned}
\mathcal{L} = \quad & \overline{l_L} i \slashed{D} l_L + \overline{e_R} i \slashed{D} e_R + \overline{\nu_R} i \slashed{\partial} \nu_R \\
& - \left( \overline{l_L} Y^e e_R \phi + \overline{l_L} Y^\nu \nu_R \phi^C + \frac{1}{2} \overline{\nu_L^C} M \nu_R + \mathrm{h.c.} \right).
\end{aligned}
\tag{1}
$$

In eq. (1) the lepton doublet $l_L$ and singlet $e_R$, $\nu_R$ fields mean those in a weak basis where, by definition, the symmetry properties are well stated. It is supposed that the ordinary chiral families of the SM with the doublet left-handed Weyl neutrinos in number $d \geq 3$ are added by the singlet Weyl neutrinos in number $s \geq 0$. Let us designate the SM extended in such a renormalizable manner as $(d, s)_{\mathrm{r}}$ extension. A priori, one should retain $s$ and $d$ as arbitrary integers, both $s \leq d$ and $s > d$ being allowed.[2] Further, $\slashed{D} \equiv \gamma^\alpha D_\alpha$ is the generic covariant derivative which reduces to the ordinary one, $\slashed{\partial} = \gamma^\alpha \partial_\alpha$, for the hypercharge zero singlet neutrinos. Here and in what follows the notations $\nu_L^C \equiv (\nu_R)^C = C \overline{\nu_R}^T$, etc, are used for the $C$

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[1]In fact, what we are talking about is lepton mixing which is described by a counterpart of the CKM matrix. But one can always choose a weak basis where the mixing matrix of charged leptons is unity. In this sense, lepton mixing is synonimous with the neutrino one.

[2]We omit the possible vector-like lepton doublets in the present analysis. Hence, with account for the most probable exclusion of the fourth heavy chiral family [9], one should put in reality $d = 3$. Nevertheless, we retain $d$ as a free parameter to better elucidate the parameter space structure of the extended SM.

conjugates of chiral fermions. $Y^e$ and $Y^\nu$ are the arbitrary complex $d \times d$ and $d \times s$ Yukawa matrices, respectively, and $M$ is a complex symmetric $s \times s$ matrix of the Majorana masses for the singlet neutrinos. Finally, $\phi$ is the Higgs isodoublet and $\phi^C \equiv i\tau_2\phi^*$ is its charge conjugate.

**Table 1** Parameter counting for the SM renormalizable extension $(d, s)_\mathrm{r}$ with $d$ doublet neutrinos and $s$ singlet ones. In this table and the ones which follow, the first and the second groups of moduli for the physical mass matrix $\mathcal{M}_{ph}$ correspond to the independent mixing angles and masses, respectively (see text).

| Couplings and symmetries | Moduli | Phases |
|---|---|---|
| $Y^e$, $Y^\nu$, $M$ | $d^2 + ds + s(s+1)/2$ | $d^2 + ds + s(s+1)/2$ |
| $G = U(d)^2 \times U(s)$ | $-d(d-1) - s(s-1)/2$ | $-d(d+1) - s(s+1)/2$ |
| $H = I$ | $0$ | $0$ |
| $\mathcal{M}_{ph}(d, s)_\mathrm{r}$ | $sd + (d+s)$ | $d(s-1)$ |
| $\mathcal{M}_{ph}(n, n)_\mathrm{r}$ | $n^2 + 2n$ | $n(n-1)$ |

The parameter counting in the weak basis for the lepton sector of the extended SM proceeds as is shown in Table 1. Here $G$ is the global symmetry of the kinetic part of the Lagrangian of eq. (1). Due to the Dirac and Majorana mass terms the symmetry $G$ is explicitly violated so that the residual symmetry is trivial, $H = I$.[3] The transformations of the broken part $G/H$ (here $G/H = G$) can be used to absorb the spurious parameters in eq. (1) leaving only the independent physical set, $\mathcal{M}_{ph}$, of them.[4] As a result, $\mathcal{M}_{ph}$ contains $sd+d+s$ independent moduli and $d(s-1)$ phases. In this, real $\mathcal{M}_{ph}$ corresponds to $CP$ conservation. Stress that only the total number of independent physical moduli is fixed by the weak basis counting. Due to absence of the left-handed Majorana masses in the weak basis, there are relations in

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[3]In what folows, we generally assume that there are no mass textures or accidental mass degeneracy. Otherwise, the residual symmetry would increase, and special consideration of each particular case would be mandatory.

[4]For this reason, parameters corresponding to symmetry $G$ are represented in tables with a minus sign, whereas those of $H$ with a plus sign.

the $(d, s)_{\mathrm{r}}$ extensions between the actual mixing angles and masses. Considering all the masses as independent ones, while a part of mixing angles as a function of the masses, would result in the formal number of mixing angles less than their actual number. This may cause some confusion in explicit parametrization. So, it is more instructive to choose all the mixing angels as independent ones, considering part of the masses as a function of the angles and the rest of the masses. To decide what is the minimal number of independent masses, consider the limit $M \to \infty$ corresponding to decoupling of $s$ heavy Majorana neutrinos. In this limit, the rest of $d$ Majorana neutrinos should become massless with necessity. Thus $d$ Majorana masses depend on $s$ ones. Clearly, it is impossible to further reduce the number of independent masses. Finally, of the physical moduli, $sd$ ones are mixing angles, the rest being masses of $d$ charged leptons and $s$ Majorana neutrinos. At $0 < s \leq d$, there are also $s$ induced Majorana masses, $d - s$ neutrinos still remaining massless.[5] At $s > d > 0$, all the $d + s$ Majorana neutrinos acquire masses. It is clear that in contrast to the quark sector, the $CP$ violation generally takes place for more than one singlet neutrino at any $d > 0$. The last line in Table 1 illustrates the extended $n$ family SM with one right-handed neutrino per family.[6,7]

In the case $(d, 0)_{\mathrm{r}}$ one has $G = U(d)^2$, all the neutrinos are massless and the residual symmetry increases up to $H = U(1)^d$ of the individual lepton numbers. Hence the number of mixing angles, as well as that of physical phases, is equal to zero.

**(ii) Only Dirac masses** There is an important case of the SM renormalizable extension. Namely, the lepton number conservation would forbid the Majorana mass terms, both the left- and right-handed. In the absence of these masses the residual symmetry at $0 < s \leq d$ would increase up to $H = U(1)$ of the total lepton number. In this case, designate it $(d, s)_{\mathrm{D}}$, $2s$ degenerate in pairs Majorana neutrinos would constitute $s$ massive Dirac ones, the rest being massless. Hence there would be $s(2d - s - 1)/2$ mixing

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[5]This reflects the fact that in this case the rank of the neutrino mass matrix is $2s$.

[6]Stress that in all the tables the number of physical masses, chosen as independent ones, is collectively that for both the charged leptons and neutrinos (Majorana or Dirac, depending on context).

[7]Our counting for the renormalizable $(d, s)_{\mathrm{r}}$ extension, both at $s \leq d$ and $s > d$, disagrees with that in ref. [6] (see remarks in section 4).

angles and $s(2d - s - 1)/2 - d + 1$ phases [7]. It follows, in particular, that at $s = d \equiv n$ for this reduced type of the $(n, n)_{\mathrm{r}}$ extension one would get $2n$ masses, $n(n-1)/2$ mixing angles and $(n-1)(n-2)/2$ phases in a complete analogy to the quark sector.

The above results are not applicable at $s > d > 0$. Here the number of massive Dirac neutrinos saturates the maximum allowed value $d$, the rest of $s - d$ Weyl neutrinos being massless. Hence the residual symmetry would increase up to $H = U(s - d) \times U(1)$, so that the number of mixing angles would be $d(d - 1)/2$ and the number of phases $(d - 1)(d - 2)/2$. The results are summarized in Table 2 along with two particular cases $(n, n - 1)_{\mathrm{D}}$ and $(n, n)_{\mathrm{D}}$ with equal numbers of mixing parameters.

**Table 2**  Parameter counting for the SM renormalizable extension $(d, s)_{\mathrm{D}}$ with only Dirac masses. The number of physical masses is that of Dirac ones.

| Couplings and symmetries | Moduli | Phases |
|---|---|---|
| $Y^e, Y^\nu$ | $d^2 + ds$ | $d^2 + ds$ |
| $G = U(d)^2 \times U(s)$ | $-d(d-1) - s(s-1)/2$ | $-d(d+1) - s(s+1)/2$ |
| $H = U(1),\ 0 < s \le d$ | $0$ | $1$ |
| $H = U(s-d) \times U(1)$ $0 < d < s$ | $(s-d)(s-d-1)/2$ | $(s-d)(s-d+1)/2 + 1$ |
| $\mathcal{M}_{ph}(d, s)_{\mathrm{D}},\ 0 < s \le d$ | $s(2d - s - 1)/2 + (d + s)$ | $s(2d - s - 1)/2 - d + 1$ |
| $\mathcal{M}_{ph}(d, s)_{\mathrm{D}},\ 0 < d < s$ | $d(d-1)/2 + 2d$ | $(d-1)(d-2)/2$ |
| $\mathcal{M}_{ph}(n, n-1)_{\mathrm{D}},\ n > 1$ | $n(n-1)/2 + (2n-1)$ | $(n-1)(n-2)/2$ |
| $\mathcal{M}_{ph}(n, n)_{\mathrm{D}},\ n > 0$ | $n(n-1)/2 + 2n$ | $(n-1)(n-2)/2$ |

# 3   Non-renormalizable extensions

**(i) Arbitrary case**  Let us now generalize the preceding considerations to the most exhaustive Dirac-Majorana case with the left-handed Majorana masses. The direct Majorana mass term for the doublet neutrinos is excluded

in the minimal SM by the symmetry and renormalizability requirements. But in the extended SM as a low energy effective theory, it could stem from the SM invariant operator of the fifth dimension

$$-\Delta\mathcal{L} = \frac{1}{2\Lambda}(\phi^{C\dagger}\tau_i\phi)(\overline{l_R^C}h\,i\tau_2\tau_i l_L) + \text{h.c.}, \tag{2}$$

with $\tau_i$, $i = 1,2,3$ being the Pauli matrices, $h$ being a $d \times d$ symmetric constant matrix, $\Lambda \gg v$ being the lepton number violating mass scale (supposedly of order of the singlet Majorana masses) and $v$ being the Higgs vacuum expectation value. The above operator with the effective isotriplet field $\Delta_i = (1/\Lambda)(\phi^{C\dagger}\tau_i\phi)$ reflects the oblique radiative corrections in the low energy Lagrangian produced by the physics beyond the SM.[8] In the unitary gauge, it yields the following mass and Yukawa term

$$-\Delta\mathcal{L} = \frac{1}{2}\Big(1 + \frac{H}{v}\Big)^2 \overline{\nu_R^C}\mu\,\nu_L + \text{h.c.}, \tag{3}$$

with $\mu = hv^2/\Lambda$.

There is no nontrivial residual symmetry in this case either, $H = I$. As for free parameters, phenomenological inclusion of such a mass term increases the numbers of moduli and phases by $d(d+1)/2$ each. Of the extra moduli, $d$ ones are the Majorana neutrino masses, the rest being physical mixing angles. Hence, the extension amounts to $d + s$ independent neutrino masses, $d(d+2s-1)/2$ physical mixing angles and the same number of phases [6]. Let us designate this general type of the SM extension as $(d, s)$, whether $s \leq d$ or $s > d$. The parameter counting for this non-renormalizable extension of the SM is summarized in Table 3.

A special case without singlet neutrinos, i.e., $(d, 0)$ extension, results in $d(d-1)/2$ mixing angles and the same number of phases [4]. Clearly, the $CP$ violation in the lepton sector becomes possible for two or more families without singlet neutrinos at all. On the other hand, the $(n, n)$ extension with $n$ complete families brings in $2n$ massive Majorana neutrinos with $n(3n-1)/2$ mixing angles and equal number of phases [5]. Hence, $CP$ violation might take place here already for one complete family.

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[8]Were the isotriplet $\Delta_i$ be considered as elementary in the framework of renormalizable extensions, it would change only the emerging Yukawa interactions not affecting the mass and mixing matrices.

**Table 3**  Parameter counting for the SM non-renormalizable extension $(d, s)$ with $d$ doublet and $s$ singlet neutrinos. Symmetries are the same as in Table 1.

| Couplings | Moduli | Phases |
|---|---|---|
| $Y^e$, $Y^\nu$, $M$ | $d^2 + ds + s(s+1)/2$ | $d^2 + ds + s(s+1)/2$ |
| $\mu$ | $+d(d+1)/2$ | $+d(d+1)/2$ |
| $\mathcal{M}_{ph}(d, s)$ | $d(d + 2s - 1)/2 + (2d + s)$ | $d(d + 2s - 1)/2$ |
| $\mathcal{M}_{ph}(n, n)$ | $n(3n - 1)/2 + 3n$ | $n(3n - 1)/2$ |

**(ii) Only Majorana masses**  Let us consider a peculiar case of the general extension above. In the absence of Yukawa couplings, $Y^\nu \equiv 0$, but at nonzero Majorana masses, both left- and right-handed, the residual symmetry is still trivial ($H = I$) as in the general case. But now the doublet and singlet neutrino sectors completely disentangle from each other. All the $d + s$ Majorana neutrinos acquire masses, and we end up with $d(d - 1)/2$ mixing angles and the same number of phases for the doublet neutrinos, without any mixing for the singlet ones, whether $s \leq d$ or $s > d$. Let us designate this case $(d, s)_{\text{M}}$. The results are collected in Table 4.[9] As for doublet neutrinos, this case formally corresponds to $(d, 0)_{\text{M}}$ which, in turn, coincides with the general one $(d, 0)$.

**Table 4**  Parameter counting for the SM extension $(d, s)_{\text{M}}$ with only Majorana masses for neutrinos. Symmetries are the same as in Table 1.

| Couplings | Moduli | Phases |
|---|---|---|
| $Y^e$, $\mu$, $M$ | $d^2 + d(d+1)/2 + s(s+1)/2$ | $d^2 + d(d+1)/2 + s(s+1)/2$ |
| $\mathcal{M}_{ph}(d, s)_{\text{M}}$ | $d(d - 1)/2 + (2d + s)$ | $d(d - 1)/2$ |

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[9]Stress that the numbers of the physical mixing angles and phases do not depend here on $s$. This is because the right-handed neutrinos are sterile in the case at hand, and their mixing matrix can be chosen to be unity in neglect of any other interactions.

# 4 Remarks

We would like to clarify some discrepancies for the renormalizable $(d, s)_\mathrm{r}$ extension between our counting in the weak basis and the one in the mass basis [6]. In the mass basis, an explicit new feature of the $(d, s)_\mathrm{r}$ extension, compared to $(d, s)$ one, is the appearance of the additional symmetry $U(d-s)$, $d \geq s$, due to $d - s$ neutrinos being massless. As a result, it is stated in the cited paper that the mixing matrix for the $(d, s)_\mathrm{r}$ extension could be obtained from the corresponding general matrix just by deleting in the latter $(d - s)^2$ spurious parameters corresponding to $U(d - s)$. We would like to remark that this procedure though being applicable does not fix the number of independent parameters and generally overestimates the number of the actual ones.

To illustrate, note that it would follow from the prescription [6], e.g., that at $d = s \equiv n$ both $(n, n)_\mathrm{r}$ and $(n, n)$ extensions would have the same numbers of mixing angles, as well as phases, $n(3n - 1)/2$, in addition to $3n$ masses. On the other hand, an arbitrary square complex matrix $Y$ can be uniquely written as a unitary matrix times a positive-definite Hermitian one, and a complex symmetric matrix $M$ can be uniquely decomposed in terms of a unitary matrix $V$ and a positive-definite diagonal one, $M = V^T M_{diag} V$. This means that with account for the global symmetry $G$ we could start in the $(n, n)_\mathrm{r}$ extension by choosing from the very beginning the Yukawa matrices $Y^e$ and $Y^\nu$ as positive-definite Hermitian matrices and $M$ as a positive definite diagonal one. As we have exhausted thus all the symmetry $G$ and there is no nontrivial residual subgroup $H$, this set of parameters is the independent physical one. It contains $n(n + 1) + n$ moduli and $n(n - 1)$ phases. This completely agrees with Table 1 and is clearly less compared to ref. [6].

We trace the origin of the discrepancy between the countings to the constraint $\mu = 0$ in eq. (3). In passing from $(d, s)$ extension to the $(d, s)_\mathrm{r}$ one, it restricts $d(d+1)/2$ phases and the same number of physical moduli. In this, $d$ of the conditions on moduli serve to determine the induced Majorana masses. Altogether, this leaves $s$ independent Majorana masses, $sd$ mixing angles and $d(s - 1)$ phases. At $0 < s \leq d$, there are $s$ induced nonzero masses, $d - s$ neutrinos remaining massless with necessity. As a consequence of the inborn masslessness for $d - s$ neutrinos, the stated constraint supersedes here those gained from the $U(d - s)$ symmetry. E.g., according to prescription [6] the

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extension $(d, 1)_r$ formally corresponds to $2d - 1$ mixing angles and $d$ phases, but explicit construction shows that there are actually just $d$ mixing angles, all of them being independent, and no phases at all. Especially clear the above constraint works at $s > d$ when there appear no massless neutrinos and there is nothing to delete by the related transformations. Nevertheless, the counting of parameters at $s > d$ for the $(d, s)_r$ extension proves to be not the same as for the $(d, s)$ one.

# 5    Conclusion

The parameter counting in the weak basis is complementary to that in the mass basis. It allows one to gain clear insight into the independent physical parameter content of the SM extensions, both renormalizable and non-renormalizable, with arbitrary numbers of the singlet and left-handed doublet neutrinos.

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